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SOME SEQUENCE SPACES OF INTERVAL NUMBERS DEFINED BY ORLICZ FUNCTION

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Abstract. In this study, we introduce some new sequence spaces of interval numbers using by Orlicz function and examine some properties of resulting sequence classes of interval numbers.

1. Introduction

The topic of interval analysis has been studied for a long time. For a detailed discussion we may suggest refer to some books, for example Moore [20]. The main issue is to regard to closed intervals as a kind of "points". Hereafter we will called them "interval numbers".

Interval arithmetic was first suggested by Dwyer [12] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [20] in 1959 and Moore and Yang [14] in 1962. Furthermore, Moore [20], Dwyer [13] and Markov [15] have developed applications to differential equations.

Chiao in [11] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Şengönül and Eryilmaz in [16] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space. Recently, Esi in [1, 2, 3, 4, 5, 6], Esi and Braha [7], Esi and Esi [8], Esi and Hazarika [9] defined and studied different properties of interval numbers.

We denote the set of all real valued closed intervals by IR. Any elements of IR is called interval number and denoted by $\overline{x} = [x_l, x_r]$. Let x_l and x_r be first and last points of x interval number, respectively. For $\overline{x}_1, \overline{x}_2 \in$I$R$, we have $\overline{x}_1 = \overline{x}_2 \Leftrightarrow x_{1_l} = x_{2_l}$, $x_{1_r} = x_{2_r}$. $\overline{x}_1 + \overline{x}_2 = \{x \in R\text{: } x_{1_l} + x_{2_l} \le x \le x_{1_r} + x_{2_r}\}$, and if $\alpha \ge 0$, then $\alpha\overline{x} = \{x \in R\text{: } \alpha x_{1_l} \le x \le \alpha x_{1_r}\}$ and if $\alpha < 0$, then $\alpha\overline{x} = \{x \in R : \alpha x_{1_r} \le x \le \alpha x_{1_l}\}$,

$$\overline{x}_1.\overline{x}_2 = \{x \in R : \min\{x_{1_l}.\ x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\} \le x \le \max\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l},\ x_{1_r}.x_{2_r}\}.$$

The set of all interval numbers IR is a complete metric space defined by

$$d(\overline{x}_1, \overline{x}_2) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\}\ [11].$$

In the special case $\overline{x}_1 = [a, a]$ and $\overline{x}_2 = [b, b]$, we obtain usual metric of R. Let us define transformation $f : N \to R$ by $k \to f(k) = \overline{x}_k$, $\overline{x} = (\overline{x}_k)$. Then $\overline{x} = (\overline{x}_k)$ is called sequence of interval numbers. The $\overline{x}_k$ is called k^{th} term of sequence $\overline{x} = (\overline{x}_k)$.

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Let w^idenotes the set of all interval numbers with real terms and the algebraic properties of w^i can be found in [15].

Now we recall the definition of convergence of interval numbers:

Definition 1.1. [11] *A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be convergent to the interval number $\overline{x}_o$ if for each $\varepsilon > 0$ there exists a positive integer k_o such that $d(\overline{x}_k, \overline{x}_o) < \varepsilon$ for all $k \geq k_o$ and we denote it by $\lim_k \overline{x}_k = \overline{x}_o$.*

Thus, $\lim_k \overline{x}_k = \overline{x}_o \Longleftrightarrow \lim_k x_{k_l} = x_{o_l}$ and $\lim_k x_{k_r} = x_{o_r}$.

The set of all closed intervals in IR is not real vector space. The main reason is that there will be no additive inverse element for each interval numbers. In this work we wish to present some special classes of interval numbers on the interval valued metric space.

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers. If $H = \sup{}_k p_k$, then for any two complex numbers a_k and b_k we have

$$|a_k + b_k|^{p_k} \leq C\left(|a_k|^{p_k} + |b_k|^{p_k}\right) \tag{1}$$

where $C = \max(1, 2^{H-1})$.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Sequence spaces defined by Orlicz functions have been investigated by Et et.al.[21], Tripathy et.al. [22], Tripathy and Dutta [23], Tripathy and Borgogain [24] and many others.

Let M be an Orlicz function, $s \geq 0$ is a reel number and $p = (p_k)$ be a sequence of positive real numbers such that $0 \leq h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. We introduce the following sequence spaces of interval number sequences.

$$\overline{\ell}_\infty(M,p,s) = \left\{\overline{x} = (\overline{x}_k) \in w^i : \sup_k k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r}\right)\right]^{p_k} < \infty, \right.$$
$$\left. s \geq 0, r > 0\right\},$$

$$\overline{c}(M,p,s) = \left\{\overline{x} = (\overline{x}_k) \in w^i : \lim_k k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k, \overline{x}_o\right)}{r}\right)\right]^{p_k} = 0, \right.$$
$$\left. s \geq 0, r > 0 \text{ for some } \overline{x}_0 \in w^i\right\},$$

$$\overline{c}_0(M,p,s) = \left\{\overline{x} = (\overline{x}_k) \in w^i : \lim_k k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r}\right)\right]^{p_k} = 0, s \geq 0, r > 0\right\},$$

and

$$\overline{\ell}(M,p,s) = \left\{\overline{x} = (\overline{x}_k) \in w^i : \sum_k k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r}\right)\right]^{p_k} < \infty, s \geq 0, r > 0\right\}.$$

2. Main Results

Theorem 2.1. *The sets $\overline{\ell}_\infty(M,p,s)$, $\overline{c}(M,p,s)$, $\overline{c}_0(M,p,s)$ and $\overline{\ell}(M,p,s)$ of sequences of interval numbers defined by a Orlicz function are closed under the coordinatewise addition and scalar multiplication.*

Proof. We consider only the set $\overline{\ell}_\infty(M,p,s)$, since it is not hard to show that the others can be treated similarly. Let define the operations "+" and "." as follows:

$$+:\overline{\ell}_\infty(M,p,s)\times\overline{\ell}_\infty(M,p,s)\to\overline{\ell}_\infty(M,p,s)$$

and

$$\cdot : R\times\overline{\ell}_\infty(M,p,s)\to\overline{\ell}_\infty(M,p,s).$$

Let $\overline{x},\overline{y}\in\overline{\ell}_\infty(M,p,s)$, then we may write

$$\sup_k k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\right]^{p_k}<\infty \text{ and } \sup_k k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{y}_k,\overline{0}\right)}{r}\right)\right]^{p_k}<\infty.$$

Since $\overline{d}(\overline{x}_k+\overline{y}_k,\overline{0})\le\overline{d}\left(\overline{x}_k,\overline{0}\right)+\overline{d}(\overline{y}_k,\overline{0})$ and using nondecreasing of M Orlicz function, we obtain

$$\begin{aligned}M(\overline{d}(\overline{x}_k+\overline{y}_k,\overline{0}))&\le M(\overline{d}\left(\overline{x}_k,\overline{0}\right)+\overline{d}(\overline{y}_k,\overline{0}))\\&\le M(\overline{d}\left(\overline{x}_k,\overline{0}\right))+M(\overline{d}(\overline{y}_k,\overline{0})).\end{aligned}$$

Since the sequence $p=(p_k)$ satisfies $0\le h=\inf_k p_k\le p_k\le\sup_k p_k=H<\infty$ and since $C=\max(1,2^{H-1})$, then

$$\begin{aligned}[M(\overline{d}(\overline{x}_k+\overline{y}_k,\overline{0}))]^{p_k}&\le[M(\ \overline{d}\left(\overline{x}_k,\overline{0}\right))+M(\overline{d}(\overline{y}_k,\overline{0}))]^{p_k}\\&\le C[M(\overline{d}\left(\overline{x}_k,\overline{0}\right))]^{p_k}+C[M(\overline{d}(\overline{y}_k,\overline{0}))]^{p_k}.\end{aligned}$$

Since (k^{-s}) is bounded, then

$$\begin{aligned}&\sup_k k^{-s}[M(\overline{d}(\overline{x}_k+\overline{y}_k,\overline{0}))]^{p_k}\\&\le\sup_k k^{-s}C[M(\overline{d}\left(\overline{x}_k,\overline{0}\right))^{p_k}]+\sup_k k^{-s}C[M(\overline{d}(\overline{y}_k,\overline{0}))]^{p_k}<\infty.\end{aligned}$$

Therefore $\overline{x}+\overline{y}\in\overline{\ell}_\infty(M,p,s)$. Now, let $\overline{x}\in\overline{\ell}_\infty(M,p,s)$. Then $\sup_k k^{-s}\left[M\left(\frac{\overline{d}(\overline{x}_k,\overline{0})}{r}\right)\right]^{p_k}<\infty$. Let $\alpha\in R$. Since $\overline{d}(\alpha\overline{x}_k,\overline{0})=|\alpha|\overline{d}(\overline{x}_k,\overline{0})$ then $M(\overline{d}(\alpha\overline{x}_k,\overline{0}))=M(|\alpha|\overline{d}(\overline{x}_k,\overline{0}))\le|\alpha|M(\overline{d}(\overline{x}_k,\overline{0}))$ and since $p=(p_k)$ is bounded sequence of positive numbers, then we obtain $[M(\overline{d}(\alpha\overline{x}_k,\overline{0}))]^{p_k}\le|\alpha|^{p_k}[M(\overline{d}(\overline{x}_k,\overline{0}))]^{p_k}$. Also (k^{-s}) is bounded, then we may write

$$\sup_k k^{-s}[M(\overline{d}(\alpha x_k,\overline{0}))]^{p_k}\le\sup_k k^{-s}|\alpha|^{p_k}[M(\overline{d}(\overline{x}_k,\overline{0}))]^{p_k}<\infty.$$

So, $\alpha\overline{x}\in\overline{\ell}_\infty(M,p,s)$. □

Theorem 2.2. *Let $p=(p_k)$ be bounded. Then classes of sequences of interval numbers $\overline{c}_0(M,p,s)$, $\overline{c}(M,p,s)$, $\overline{\ell}_\infty(M,p,s)$ are complete metric spaces with respect to the following metric*

$$\overline{d}_\infty(\overline{x},\overline{y})=\inf\left\{r^{\frac{p_k}{T}}:\sup_k k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k,\overline{y}_k\right)}{r}\right)\right]^{p_k}\le 1\right\}$$

and the class of sequences of interval numbers $\overline{\overline{\ell}}(M,p,s)$ is complete metric space with respect to the following metric

$$\overline{d}_p(\overline{x},\overline{y})=\inf\left\{r^{\frac{p_k}{T}}:\sum_k k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k,\overline{y}_k\right)}{r}\right)\right]^{p_k}\le 1\right\}$$

where $T=\max(1,\sup_k p_k=H<\infty)$.

Proof. It can be easily verified that the classes are metric spaces. To prove completeness we consider only $\overline{c}_0(M,p,s)$. Others can be treated similarly. Let $(\overline{x}^i) = \{\overline{x}_0^i, \overline{x}_1^i, \dots\}$ be a Cauchy sequence in $\overline{c}_0(M,p,s)$. Then $\overline{d}_\infty(\overline{x}^i, \overline{x}^j) \to 0$ as $i,j \to \infty$. For given $\varepsilon > 0$ choose $r > 0$ and $x_o > 0$ be such that $\frac{\varepsilon}{rx_0} > 0$ and $M\left(\frac{rx_0}{2}\right) \geq 1$. Then there exists $n_o \in N$ such that $\overline{d}_\infty(\overline{x}^i, \overline{x}^j) < \frac{\varepsilon}{rx_0}$ for all $i,j \geq n_o$. This implies

$$\inf\left\{ r^{\frac{p_k}{T}} : \sup_k k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k^i, \overline{x}_k^j\right)}{r}\right)\right]^{p_k} \leq 1 \right\} < \frac{\varepsilon}{rx_0}.$$

Now, $M\left(\frac{\overline{d}(\overline{x}_k^i, \overline{x}_k^j)}{r}\right) \leq 1 \leq M(\frac{rx_0}{2})$ implies that $\frac{\overline{d}(\overline{x}_k^i, \overline{x}_k^j)}{\overline{d}_\infty(\overline{x}^i, \overline{x}^j)} \leq \frac{rx_0}{2}$. So we obtain $\overline{d}\left(\overline{x}_k^i, \overline{x}_k^j\right) < \frac{rx_0}{2}\frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2}$. This implies $\left(\overline{x}_k^{(i)}\right)$ is a Cauchy sequence of interval numbers in IR for all fixed $k \in N$. Since the set of interval numbers set IR is complete, so there exists an interval number $\overline{x} = (\overline{x}_k)$, such that $\overline{x}_k^i \to \overline{x}_k$ as i$\to \infty$. Now

$$\lim_{j\to\infty} \sup_k k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k^i, \overline{x}_k^j\right)}{r}\right)\right] \leq 1 \Longrightarrow \sup_k k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k^i, \overline{x}_k\right)}{r}\right)\right] \leq 1.$$

Let $j \geq n_o$ then taking infimum of such r, we have $\overline{d}\left(\overline{x}_k^i, \overline{x}_k\right) < \varepsilon$. Now using $\overline{d}\left(\overline{x}_k, \overline{0}\right) \leq \overline{d}\left(\overline{x}_k, \overline{x}_k^i\right) + \overline{d}\left(\overline{x}_k^i, \overline{0}\right)$ we get $\overline{x} = (\overline{x}_k) \in \overline{c}_0(M,p,s)$. Since $\{\overline{x}_k^{(i)}\}$ is arbitrary Cauchy sequence then the space $\overline{c}_0(M,p,s)$ is complete. □

Theorem 2.3. *Let* $\inf_k p_k = h > 0$. *Then*

a) $\overline{x} = (\overline{x}_k) \in \overline{c}$ *implies* $\overline{x} = (\overline{x}_k) \in \overline{c}(M,p,s)$, ***b)*** $\overline{x} = (\overline{x}_k) \in \overline{c}(p,s)$ *implies* $\overline{x} = (\overline{x}_k) \in \overline{c}(M,p,s)$.

Proof. **a)** Suppose that $\overline{x} = (\overline{x}_k) \in \overline{c}$. Then there is at least one $\overline{x}_0 \in \overline{c}$ such that $\lim_{k\to\infty} \overline{d}(\overline{x}_k, \overline{x}_0) = 0$. As M is Orlicz function, then

$$\lim_{k\to\infty}[M(\overline{d}(\overline{x}_k, \overline{x}_0))] = M[\lim_{k\to\infty} \overline{d}(\overline{x}_k, \overline{x}_0)] = M(0) = 0.$$

As $\inf_k p_k = h > 0$ then $\lim_{k\to\infty}[M(\overline{d}(\overline{x}_k, \overline{x}_0))]^h = 0$. So for $0 < \varepsilon < 1$, $\exists k_0 \in N$ such that for all $k > k_0$, $[M(\overline{d}(\overline{x}_k, \overline{x}_0))]^h < \varepsilon < 1$ and since $p_k \geq h$, we may write $[M(\overline{d}(\overline{x}_k, \overline{x}_0))]^{p_k} \leq [M(\overline{d}(\overline{x}_k, \overline{x}_0))]^h < \varepsilon$ for all k. Then we obtain $\lim_{k\to\infty}[M(\overline{d}(\overline{x}_k, \overline{x}_0))]^{p_k} = 0$. As (k^{-s}) is bounded, we can write $\lim_{k\to\infty} k^{-s}[M(\overline{d}(\overline{x}_k, \overline{x}_0))]^{p_k} = 0$. Therefore $\overline{x} = (\overline{x}_k) \in \overline{c}(M,p,s)$.

b) Let $\overline{x} = (\overline{x}_k) \in \overline{c}(p,s)$, then $a_k = k^{-s}[\overline{d}(\overline{x}_k, \overline{x}_0)]^{p_k} \to 0$ as $k \to \infty$. Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \leq t \leq \delta$. Now we write

$$I_1 = \{k \in N : \overline{d}(\overline{x}_k, \overline{x}_0) \leq \delta\} \text{ and } I_2 = \{k \in N : \overline{d}(\overline{x}_k, \overline{x}_0) > \delta\}.$$

For $k \in I_2$ and $\overline{d}(\overline{x}_k, \overline{x}_0) > \delta$

$$\overline{d}\left(\overline{x}_k, \overline{x}_0\right) < \overline{d}\left(\overline{x}_k, \overline{x}_0\right)\delta^{-1} < 1 + \left[\overline{d}\left(\overline{x}_k, \overline{x}_0\right)\delta^{-1}\right]$$

where $[t]$ denotes the integer part of t. By using properties of Orlicz function M, for $\overline{d}(\overline{x}_k, \overline{x}_0) > \delta$, we have

$$M(\overline{d}(\overline{x}_k, \overline{x}_0)) \leq (1 + \left[\overline{d}\left(\overline{x}_k, \overline{x}_0\right).\delta^{-1}\right])M(1) \leq 2M(1)\ \overline{d}\left(\overline{x}_k, \overline{x}_0\right)\delta^{-1}$$

and for $k \in I_1$ and $\overline{d}(\overline{x}_k, \overline{x}_0) \leq \delta \Longrightarrow M(\overline{d}\,(\overline{x}_k, \overline{x}_0)) < \varepsilon$. Hence

$$\begin{aligned} k^{-s}[M(\overline{d}(\overline{x}_k, \overline{x}_0))]^{p_k} &= k^{-s}[M(\overline{d}(\overline{x}_k, \overline{x}_0))]^{p_k}{}_{I_1} + k^{-s}[M(\overline{d}(x, \overline{x}_0))]^{p_k}{}_{I_2} \\ &\leq k^{-s}[M(\overline{d}(\overline{x}_k, \overline{x}_0))]^{p_k} \\ &< k^{-s}\varepsilon^H + [2M(1)\delta^{-1}]^H.a_k \to 0 \text{ as } k \to \infty. \end{aligned}$$

Then we obtain $\overline{x} = (\overline{x}_k) \in \overline{c}(M, p, s)$. This completes the proof. □

Theorem 2.4. *Let M and N be two Orlicz functions and s, s_1, $s_2 \geq 0$ be real numbers. Then*
a) $\overline{c}(M, p, s) \cap \overline{c}(N, p, s) \subset \overline{c}(M + N, p, s)$, ***b)*** *If $s_1 \leq s_2$ then $\overline{c}(M, p, s_1) \subset \overline{c}(M, p, s_2)$.*

Proof. **a)** Let $\overline{x} = (\overline{x}_k) \in \overline{c}(M, p, s) \cap \overline{c}(N, p, s)$. From (1), we have

$$\begin{aligned} [(M + N)(\overline{d}\,(\overline{x}_k, \overline{x}_0))]^{p_k} &= [M\left(\overline{d}\,(\overline{x}_k, \overline{x}_0)\right) + N\left(\overline{d}\,(\overline{x}_k, \overline{x}_0)\right)]^{p_k} \\ &\leq C[M(\overline{d}\,(\overline{x}_k, \overline{x}_0))]^{p_k} + C[N(\overline{d}\,(\overline{x}_k, \overline{x}_0))]^{p_k}. \end{aligned}$$

As (k^{-s}) is bounded, we can write

$$k^{-s}[(M + N)(\overline{d}\,(\overline{x}_k, \overline{x}_0))]^{p_k} \leq Ck^{-s}[M(\overline{d}\,(\overline{x}_k, \overline{x}_0))]^{p_k} + Ck^{-s}[N(\overline{d}\,(\overline{x}_k, \overline{x}_0))]^{p_k}.$$

Hence we obtain $\overline{x} = (\overline{x}_k) \in \overline{c}(M + N, p, s)$. □

Theorem 2.5. *Let M be an Orlicz function, then*
a) $\overline{\ell}_\infty \subset \overline{\ell}_\infty(M, p, s)$, ***b)*** *If M is bounded then $\overline{\ell}_\infty(M, p, s) = w^i$.*

Proof. **a)** *Let* $\overline{x} = (\overline{x}_k) \in \overline{\ell}_\infty$. Then there exists a positive integer $G \geq 0$ such that $\overline{d}\left(\overline{x}_k, \overline{0}\right) \leq G$. Then the sequence $(M(\overline{d}\left(\overline{x}_k, \overline{0}\right)))$ is also bounded. Hence

$$k^{-s}[M(\overline{d}(\overline{x}_k, \overline{0}))]^{p_k} \leq k^{-s}[GM(1)]^{p_k} \leq k^{-s}[GM(1)]^H < \infty.$$

Therefore $\overline{x} = (\overline{x}_k) \in \overline{\ell}_\infty(M, p, s)$.
b) If M Orlicz function is bounded then for any $\overline{x} = (\overline{x}_k) \in w^i$, $k^{-s}[M(\overline{d}(\overline{x}_k, \overline{0}))]^{p_k} \leq k^{-s}L^{p_k} \leq k^{-s}L^H < \infty$. Hence we obtain $\overline{\ell}_\infty(f, p, s) = w^i$. □

Theorem 2.6. *The spaces $\overline{\ell}_\infty(p)$, $\overline{c}_0(p)$ and $\overline{\ell}(p)$ are solid spaces.*

Proof. Let $\bar{\lambda}$ denotes the anyone of the spaces $\overline{\ell}_\infty(p)$, $\overline{c}_0(p)$ and $\overline{\ell}(p)$. Suppose that $\overline{d}\left(\overline{y}_k, \overline{0}\right) \leq \overline{d}(\overline{x}_k, \overline{0})$ holds, for some $\overline{x} = (\overline{x}_k) \in \bar{\lambda}$. Therefore, one can easily see that $\sup_k[\overline{d}\left(\overline{y}_k, \overline{0}\right)]^{p_k} \leq \sup_k[\overline{d}\left(\overline{x}_k, \overline{0}\right)]^{p_k} < \infty$, $\lim_k[\overline{d}\left(\overline{y}_k, \overline{0}\right)]^{p_k} \leq \lim_k\left[\overline{d}\left(\overline{x}_k, \overline{0}\right)\right]^{p_k} = 0$ and $\sum_k\left[\overline{d}(\overline{y}_k, \overline{0})\right]^{p_k} \leq \sum_k\left[\overline{d}(\overline{x}_k, \overline{0})\right]^{p_k} < \infty$. These inequalities yield to desired consequence that $y = (\overline{y}_k) \in \bar{\lambda}$. □

Theorem 2.7. *The spaces $\overline{\ell}_\infty(M, p, s)$, $\overline{c}_0(M, p, s)$ and $\overline{\ell}(M, p, s)$ are solid spaces.*

Proof. This is immediate by Teorem 2.6 and since the Orlicz function M is increasing. □

Theorem 2.8. *Let M be an Orlicz function.*
a) *If $0 < \inf p_k = h \leq p_k \leq 1$, then $\overline{c}(M, p, s) \subset \overline{c}(M, s)$,* ***b)*** *If $1 \leq p_k \leq \sup p_k < \infty$, then $\overline{c}(M, s) \subset \overline{c}(M, p, s)$,* ***c)*** *Let $0 < p_k \leq q_k$ for each $k \in \mathbb{N}$. Then we have $\overline{c}(M, p, s) \subset \overline{c}(M, q, s)$.*

Proof. **a)** The proof is obtained by using the following inequality:

$$k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k,\overline{x}_0\right)}{r}\right)\right] \leq k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k,\overline{x}_0\right)}{r}\right)\right]^{p_k}.$$

b) The proof is obtained from the following inequality:

$$k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k,\overline{x}_0\right)}{r}\right)\right]^{p_k} \leq k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k,\overline{x}_0\right)}{r}\right)\right].$$

c) Let $\overline{x} = (\overline{x}_k) \in \overline{c}(M,p,s)$, that is $k^{-s}\left[M\left(\frac{\overline{d}(\overline{x}_k,\overline{x}_0)}{r}\right)\right]^{p_k} = 0$.

This implies that $k^{-s}\left[M\left(\frac{\overline{d}(\overline{x}_k,\overline{x}_0)}{r}\right)\right]^{p_k} \leq 1$ for sufficiently large k. Since M Orlicz function is non-decreasing we have

$$k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k,\overline{x}_0\right)}{r}\right)\right]^{q_k} \leq k^{-s}\left[M\left(\frac{\overline{d}\left(\overline{x}_k,\overline{x}_0\right)}{r}\right)\right]^{p_k} = 0,$$

i.e., $\overline{x} = (\overline{x}_k) \in \overline{c}(M,p,s)$. This completes the proof.

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